

# METRIC PROPERTIES OF MANIFOLDS BIMEROMORPHIC TO COMPACT KÄHLER SPACES

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## Introduction

A goal of this paper is to prove that: "Every compact complex manifold  $M$  bimeromorphic to a compact Kähler manifold  $M'$  is balanced; that is,  $M$  has a hermitian metric with Kähler form  $\omega$  such that  $d\omega^{N-1} = 0$ ,  $N = \dim M$ " (Corollary 4.5). Of course, every Kähler manifold is balanced; the interest of the above result stems from the fact that we find out a metric property which transfers from  $M'$  to  $M$ , while it is well known that the Kähler property is not stable under bimeromorphic maps.

This introduction is mainly devoted to outline the background.

Let  $M$  and  $\tilde{M}$  be compact complex manifolds and  $f: \tilde{M} \rightarrow M$  be a modification. It is well known that:

- (1) If  $f$  is a blow-up of  $M$  with smooth center and  $M$  is Kähler, then  $\tilde{M}$  is Kähler too [4],

however

- (2) in general, if  $f$  is a modification and  $M$  is Kähler,  $\tilde{M}$  fails to be Kähler.

A counterexample is given in [12, p. 505] by a compact non-Kähler threefold  $X$  and a modification  $f: X \rightarrow \mathbf{P}_3$ . In order to illustrate Chow's lemma, Hironaka builds up also a projective threefold  $Y$  and a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{g} & X \\ & \searrow h & \downarrow f \\ & & \mathbf{P}_3 \end{array}$$

where  $g$  and  $h$  are obtained as a finite sequence of blow-ups with smooth centers.

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Received September 26, 1991. This work is partially (40%) supported by Ministero della Università della Ricerca Scientifica e Tecnologica.

Let us consider for a moment the threefold  $X$ . Since a compact Kähler manifold cannot contain any complex curve homologous to zero, but  $X$  contains such a curve (see [17, Chapter VIII,3.3.] or [9, p. 444]), it is not Kähler. On the other hand, a compact balanced manifold contains no hypersurfaces homologous to zero and neither does  $X$  by construction: so  $X$  could be balanced. This property of compact balanced manifolds has a weak converse, if you look at hypersurfaces as positive currents of degree  $(1,1)$ . Indeed Michelson proved that a compact complex manifold is balanced if and only if it carries no positive currents of degree  $(1,1)$  which are components of a boundary [16, Proposition 4.5]. This result suggested that  $X$  is balanced, as we proved in [1]; yet it is only a particular case of the following general statement:

(3) If  $M$  is Kähler, then  $\widetilde{M}$  is balanced [2].

Balanced manifolds have been studied from a differential point of view in [6]; other results and examples can be found of course in [16]. (3) shows how balanced manifolds can be “produced” in a very natural way by using modifications. Besides, we can also “pull back” the property of having a balanced metric (not the balanced metric itself, in general!) as is shown in [3]:

(3') If  $M$  is balanced,  $\widetilde{M}$  is balanced too.

(3') proves that going from  $M$  to  $\widetilde{M}$  via  $f$  no new hypersurface (and also positive current) which is the component of a boundary can appear (on the contrary, new curves may appear as  $f: X \rightarrow \mathbf{P}_3$  shows).

Thus the following question arises naturally: Is the class of compact balanced manifolds invariant by modifications? In other words, can statement (3') be reversed: If  $\widetilde{M}$  is balanced, is  $M$  balanced too? The problem looks interesting even in the simplest case because, as the modification  $g: Y \rightarrow X$  shows,

(4) even if  $f$  is a blow-up of  $M$  with smooth center and  $\widetilde{M}$  is Kähler, in general  $M$  fails to be Kähler.

In this paper we prove that for a generic modification  $f: \widetilde{M} \rightarrow M$

(5) if  $\widetilde{M}$  is Kähler, then  $M$  is balanced.

The proof of (5) depends heavily on the quoted result of [16] and on our Theorem 3.9: “Suppose  $M$  and  $\widetilde{M}$  are complex manifolds (not necessarily compact) and  $f: \widetilde{M} \rightarrow M$  is a proper modification. If  $T$  is a positive

$\partial\bar{\partial}$ -closed current on  $M$  of degree  $(1, 1)$ , then there exists a positive  $\partial\bar{\partial}$ -closed current  $\tilde{T}$  on  $\tilde{M}$  of degree  $(1, 1)$  such that  $f_*\tilde{T} = T$ . Moreover if  $\tilde{M}$  is compact, such a current is unique.”

Since locally  $\partial\bar{\partial}$ -closed currents are components of a boundary, it is convenient to translate the condition in [16] in terms of Aeppli cohomology groups (see (1.3)). To generalize (5), we shall introduce a cohomological condition, called (B), which holds in particular for compact Kähler manifolds. In Theorem 3.9, if  $\tilde{M}$  satisfies (B), the cohomology class  $[\tilde{T}]$  of  $\tilde{T}$  is exactly  $f^*[T]$  (see Proposition 3.10). So we get the following converse of (3') :

(5') (Main Theorem 4.2) If  $\tilde{M}$  is balanced and satisfies (B), then  $M$  is balanced and satisfies (B) too.

Now Corollary 4.5 announced at the beginning is simply a consequence of (5) and of a theorem of Varouchas [22].

As one can see in the literature, the most interesting case is that of compact complex manifolds which are bimeromorphic to projective varieties, that is, Moishezon manifolds. Namely, let  $M$  be a Moishezon manifold: If  $\tilde{M}$  is projective and  $f: \tilde{M} \rightarrow M$  is a modification, it is difficult to find smooth objects on  $M$  coming from  $\tilde{M}$ . For instance, if  $\tilde{\omega}$  is a Kähler form on  $\tilde{M}$ , then  $f_*\tilde{\omega}$  is not smooth: its coefficients are in  $L^1_{loc}$ . Moreover, if  $L$  is a positive line bundle on  $\tilde{M}$ , although  $f_*L$  is a holomorphic line bundle on the whole of  $M$  [18], it is not, in general, positive (for a survey see [23]).

Therefore our techniques based on positive,  $\partial\bar{\partial}$ -closed currents seem to be more appropriate and allow us to assert that every Moishezon manifold carries a balanced metric.

Finally notice that Michelson's characterization theorem is not constructive, therefore if  $\tilde{M}$  and  $M$  are balanced, the results (3') and (5') do not give any information about the link between balanced metrics on  $\tilde{M}$  and on  $M$ . Nevertheless, we shall prove that: "For every balanced metric  $h$  on  $M$  with Kähler form  $\omega$  there exists a balanced metric  $\tilde{h}$  on  $\tilde{M}$  with Kähler form  $\tilde{\omega}$  such that  $\omega^{N-1} - f_*\tilde{\omega}^{N-1}$  is a  $\partial\bar{\partial}$ -exact current." This is a corollary of Theorem 4.8: "Let  $M$  and  $\tilde{M}$  be  $p$ -Kähler manifolds, let  $f: \tilde{M} \rightarrow M$  be a proper modification and call  $Y$  the degeneracy set, with  $p > \dim Y$ . For every  $p$ -Kähler form  $\Omega$  on  $M$ , there exists a  $p$ -Kähler form  $\tilde{\Omega}$  on  $\tilde{M}$  such that  $\Omega - f_*\tilde{\Omega}$  is a  $\partial\bar{\partial}$ -exact current."

### 1. Preliminaries and notation

(1.1) Throughout the paper, whether explicitly stated or not,  $\widetilde{M}$  and  $M$  are assumed to be complex  $N$ -dimensional manifolds. A *proper modification*  $f: \widetilde{M} \rightarrow M$  is a proper holomorphic map such that, for a suitable analytic set  $Y$  in  $M$ ,  $E := f^{-1}(Y)$  (the *exceptional set of the modification*) is a hypersurface and  $\widetilde{M} - E \xrightarrow{f} M - Y$  is a biholomorphism. Moreover,  $Y$  has codimension  $\geq 2$  or  $f$  is a biholomorphism.

In particular, if  $Y$  is smooth and  $f$  is the blow-up of  $M$  along  $Y$ , suitable coordinates can be chosen in  $M$  as follows. (As usual,  $B_k(z^\circ, r)$  denotes the euclidean open ball in  $\mathbf{C}^k$  with center  $z^\circ$  and radius  $r$ .  $B_k(0, r)$  is simply denoted by  $B_k(r)$  and  $B_k(1)$  is denoted by  $B_k$ . Take  $B_0 := \{0\}$ .) For every  $y \in Y$  take an open neighborhood  $U = B_m \times B_n$  ( $m := \dim Y$  and  $N := m + n$ ) such that  $U \cap Y = B_m \times \{0\}$ . Call  $\widetilde{U} := f^{-1}(U)$ : We can identify

$$f|_{\widetilde{U}}: \widetilde{U} \rightarrow U \quad \text{with the natural projection } \pi: B_m \times \widetilde{B}_n \rightarrow B_m \times B_n,$$

where  $\widetilde{B}_n$  denotes the blow-up of  $B_n$  at  $\{0\}$ . In the following we shall simply say: "identify locally  $f$  with  $\pi$ ."

Recall that if  $(t_1, \dots, t_m) \in \mathbf{C}^m$  and  $(z_1, \dots, z_n) \in \mathbf{C}^n$ ,

$$\widetilde{B}_n = \{(z, \xi) \in B_n \times \mathbf{P}^{n-1} \mid z_j \xi_k = z_k \xi_j, 1 \leq j, k \leq n\}$$

so that the natural Kähler form on  $\widetilde{U}$  is given by

$$\tilde{\omega} := \frac{i}{2} \partial \bar{\partial} \|t\|^2 + \frac{i}{2} \partial \bar{\partial} \|z\|^2 + \frac{i}{2\pi} \partial \bar{\partial} \log \|\xi\|^2;$$

therefore (see [23, p. 37])

$$\pi_* \tilde{\omega} = \frac{i}{2} \partial \bar{\partial} \|t\|^2 + \frac{i}{2} \partial \bar{\partial} \|z\|^2 + \frac{i}{2\pi} \partial \bar{\partial} \log \|z\|^2.$$

(1.2) As usual,  $\mathcal{E}^{k,k}(M)_{\mathbf{R}}$  (resp.  $\mathcal{D}^{k,k}(M)_{\mathbf{R}}$ ) denotes the space of smooth real  $(k, k)$ -forms (resp. smooth real  $(k, k)$ -forms with compact support) on  $M$ . Their duals are spaces of real currents of bidimension  $(k, k)$  (or of degree  $(N-k, N-k)$ ), i.e., real  $(N-k, N-k)$ -forms with distribution coefficients.

Let  $\varphi \in \mathcal{E}^{k,k}(M)_{\mathbf{R}}$ . In local coordinates, we shall often write

$$\begin{aligned} \varphi &= \frac{ik^2}{2^k} \sum_{1 \leq \alpha_1 < \dots < \alpha_k \leq N, 1 \leq \beta_1 < \dots < \beta_k \leq N} \varphi_{\alpha_1, \dots, \alpha_k \bar{\beta}_1, \dots, \bar{\beta}_k}(z) dz_{\alpha_1} \\ &\quad \wedge \dots \wedge dz_{\alpha_k} \wedge d\bar{z}_{\beta_1} \wedge \dots \wedge d\bar{z}_{\beta_k} \\ &= \sigma_k \sum_{|A|=|B|=k} \varphi_{A\bar{B}} dz_A \wedge d\bar{z}_B, \end{aligned}$$

where  $\sum'$  denotes the sum on strictly increasing multi-indices.

A real current  $T$  on  $M$  of bidimension  $(k, k)$  is called *positive* (in the sense of Lelong [15]) if, for every choice of  $\varphi_1, \dots, \varphi_k \in \mathcal{D}^{1,0}(M)_{\mathbf{R}}$ ,  $T(\sigma_k \varphi_1 \wedge \dots \wedge \varphi_k \wedge \bar{\varphi}_1 \wedge \dots \wedge \bar{\varphi}_k) \geq 0$ . Moreover  $T$  is said to be *strictly positive* if  $\varphi_1 \wedge \dots \wedge \varphi_k \neq 0$  implies  $T(\sigma_k \varphi_1 \wedge \dots \wedge \varphi_k \wedge \bar{\varphi}_1 \wedge \dots \wedge \bar{\varphi}_k) > 0$ . It is well known that a positive current is of order zero. We shall denote by  $\|T\|$  the mass of  $T$ . A smooth form  $\psi \in \mathcal{E}^{N-k, N-k}(M)_{\mathbf{R}}$  is *positive* (resp. *strictly positive*) if the associated current  $T_\psi$ , defined as

$$T_\psi(\varphi) = \int \varphi \wedge \psi \quad \forall \varphi \in \mathcal{D}^{k,k}(M)_{\mathbf{R}},$$

is a positive (resp. strictly positive) current.

If  $X$  is a  $p$ -dimensional irreducible analytic subset of  $M$ , we shall denote by  $[X]$  the positive current defined as

$$[X](\varphi) := \int_X \varphi \quad \forall \varphi \in \mathcal{D}^{p,p}(M)_{\mathbf{R}}.$$

It is well known that  $[X]$  is closed; moreover, if  $u: X \rightarrow \mathbf{R}$  is a pluriharmonic function,  $u[X]$  is a  $\partial\bar{\partial}$ -closed current.

(1.3) As regards the statement of the Main Theorem, we recall here the definition of balanced manifold and define condition (B).

**1.1. Definition.** Let  $M$  be a complex  $N$ -dimensional manifold.  $M$  is said to be *balanced* (or semi-Kähler) if there exists a hermitian metric  $h$  on  $M$ , called the *balanced metric*, such that its Kähler form  $\omega$  satisfies  $d\omega^{N-1} = 0$ .

This class of manifolds obviously includes that of Kähler manifolds (for  $N = 2$  they coincide) but also many important classes of non-Kähler manifolds, such as the complex solvmanifolds, twistor spaces of oriented riemannian 4-manifolds, 1-dimensional families of Kähler manifolds (see [16]), hermitian compact manifolds which are locally flat [6], manifolds obtained as modifications of compact Kähler manifolds [2]. As well as in the Kähler case (see [11]), there exists an intrinsic characterization of compact balanced manifolds by means of positive currents.

**1.2. Theorem** [16, Theorem 4.5]. *Suppose  $M$  is a compact complex manifold. The following conditions are equivalent:*

- (i)  $M$  is balanced.
- (ii) If  $T$  is a positive current on  $M$  of degree  $(1, 1)$  which is the component of a boundary (i.e., there exists a current  $S$  such that  $T = \bar{\partial}S + \partial\bar{S}$ ), then  $T = 0$ .

Let us say a few words on currents which are components of boundaries. If  $T$  is a current of bidimension  $(N-1, N-1)$  which is the component of

a boundary, or, more generally, which is a (weak) limit of currents which are components of boundaries, then  $\partial\bar{\partial}T = 0$  and moreover  $T(\varphi) = 0$  for every closed  $\varphi \in \mathcal{D}^{N-1, N-1}(M)_{\mathbf{R}}$ ; that is, if we consider the operator

$$d: \mathcal{D}^{N-1, N-1}(M)_{\mathbf{R}} \rightarrow (\mathcal{D}^{N, N-1}(M) \oplus \mathcal{D}^{N-1, N}(M))_{\mathbf{R}}$$

and its dual

$$(\partial \oplus \bar{\partial}): (\mathcal{D}^{N, N-1}(M) \oplus \mathcal{D}^{N-1, N}(M))_{\mathbf{R}}' \rightarrow (\mathcal{D}^{N-1, N-1}(M)_{\mathbf{R}})',$$

then  $(\text{Ker } d)^{\perp} = \overline{\text{Im}(\partial \oplus \bar{\partial})}$ .

In [16, Lemma 4.8] it is proved that, if  $M$  is compact,  $\text{Im}(\partial \oplus \bar{\partial})$  is a closed subspace of  $\text{Ker } i\partial\bar{\partial}$ ; hence every current which is the limit of currents which are components of boundaries is the component of a boundary itself. Nevertheless, we shall work mainly in the noncompact case.

The *real (1, 1)-Aeppli groups* are defined as follows:

$$V^{1,1}(M)_{\mathbf{R}} = \frac{\text{Ker}(i\partial\bar{\partial}: \mathcal{E}^{1,1}(M)_{\mathbf{R}} \rightarrow \mathcal{E}^{2,2}(M)_{\mathbf{R}})}{(\partial\mathcal{E}^{0,1}(M) + \bar{\partial}\mathcal{E}^{1,0}(M))_{\mathbf{R}}}$$

and

$$\Lambda^{1,1}(M)_{\mathbf{R}} = \frac{\text{Ker}(d: \mathcal{E}^{1,1}(M)_{\mathbf{R}} \rightarrow (\mathcal{E}^{2,1}(M) + \mathcal{E}^{1,2}(M))_{\mathbf{R}})}{i\partial\bar{\partial}\mathcal{E}^{0,0}(M)_{\mathbf{R}}}.$$

As usual, we shall denote by  $H^{1,1}(M, \mathbf{R})$  the set of classes in  $H^2(M, \mathbf{R})$  which have a (1, 1)-representative. It is well known that all these groups can be defined also by means of real currents of degree (1, 1). Thus a  $\partial\bar{\partial}$ -closed current  $T$  is the component of a boundary if and only if its class in  $V^{1,1}(M)_{\mathbf{R}}$  is zero.

A class of  $V^{1,1}(M)_{\mathbf{R}}$  is said to be *positive* if it can be represented by a positive current: hence Theorem 1.2 can be written as follows: “ $M$  is balanced if and only if every nonzero positive  $\partial\bar{\partial}$ -closed current of degree (1, 1) represents a nonzero class in  $V^{1,1}(M)_{\mathbf{R}}$ .” Finally, let us consider the natural maps:

$$\begin{aligned} \alpha: \Lambda^{1,1}(M)_{\mathbf{R}} &\rightarrow H^{1,1}(M, \mathbf{R}), \\ \beta: H^{1,1}(M, \mathbf{R}) &\rightarrow V^{1,1}(M)_{\mathbf{R}} \end{aligned}$$

and the following condition:

- (B)  $\beta$  is injective and  $\text{Im } \beta$  contains all positive elements of  $V^{1,1}(M)_{\mathbf{R}}$ .

**1.3. Proposition.** *If  $\beta \circ \alpha: \Lambda^{1,1}(M, \mathbf{R})_{\mathbf{R}} \rightarrow V^{1,1}(M)_{\mathbf{R}}$  is an isomorphism, then  $\alpha$  and  $\beta$  are isomorphisms. In particular, every compact Kähler manifold satisfies (B).*

*Proof.* It is enough to notice that  $\alpha$  is always surjective. Moreover, if  $M$  is regular (in particular, if it is Kähler or Moishezon or in the class  $\mathcal{E}$  of Fujiki), then  $\beta \circ \alpha$  is an isomorphism (see [21] for the definition of regular manifold and its cohomological properties).

**2.  $\partial\bar{\partial}$ -closed currents and pluriharmonic functions**

We study in this section the behaviour of real  $\partial\bar{\partial}$ -closed currents of degree  $(1, 1)$  and of order zero, whose support is contained in the exceptional set of a proper modification. If the support is “too small” (that is, it is contained in an analytic subset of dimension  $< N - 1$ ), the current vanishes (see Theorem 2.1; if  $T$  is also positive see [2, Theorem 1.5]). On the other hand, if the modification  $f: \widetilde{M} \rightarrow M$  is obtained as a finite sequence of blow-ups with smooth centers, we get a characterization of the set of currents described above: “Every real  $\partial\bar{\partial}$ -closed current  $T$  of degree  $(1, 1)$  and order zero supported in the exceptional set  $E$  of  $f$  is of the form  $\sum_{\alpha} u_{\alpha}[E_{\alpha}]$ , where  $\{E_{\alpha}\}$  is the set of irreducible components of  $E$  and  $u_{\alpha}$  is a pluriharmonic function on  $E_{\alpha}$ ” (Proposition 2.5). These results are well known for locally flat currents, but we are not in this case, as Remark 2.4 shows.

We get moreover that if such a current is limit of currents which are components of boundaries, then it vanishes. This holds also in a weaker form if  $f$  is a generic proper modification (Proposition 2.7).

**2.1. Theorem.** *Let  $\Omega$  be an open set in  $\mathbf{C}^N$ , and suppose  $T$  is a real current of bidimension  $(p, p)$  and of order zero on  $\Omega$  such that  $\partial\bar{\partial}T = 0$ . If the support of  $T$  is contained in an analytic subset  $Y$  of  $\Omega$  of dimension  $q < p$ , then  $T = 0$ .*

*Proof.* Let  $x \in \text{Reg } Y$ ; in a neighborhood  $U$  of  $x$  choose coordinates  $(z_1, \dots, z_N)$  such that

$$Y \cap U = \{z \in U \mid z_j = 0 \text{ for } j = q + 1, \dots, N\}.$$

Call  $z' = (z_1, \dots, z_q)$  and  $z'' = (z_{q+1}, \dots, z_N)$ . In  $U$ ,

$$T = \sigma_{N-p} \sum'_{|A|=|B|=N-p} T_{A\bar{B}} d z_A \wedge d \bar{z}_B$$

where the measures  $t_{A\bar{B}}$  can be written as

$$t_{A\bar{B}}(z) = r_{A\bar{B}}(z') \otimes \delta(z'')$$

because  $\text{supp } T \subset Y$  (see for instance [14], p. 47).

Call  $I = \{q + 1, \dots, N\}$ : Since  $q < p$ ,  $A \supseteq I$  and  $B \supseteq I$  for all strictly increasing  $(N - p)$ -indices  $A$  and  $B$ . Choose  $\alpha \in I \setminus A$ ,  $\beta \in I \setminus B$  and let  $A' = A \cup \{\alpha\}$ ,  $B' = B \cup \{\beta\}$  (arrange indices in increasing order). Compute  $i\partial\bar{\partial}T$ , and notice that, in the coefficient of  $dz_{A'} \wedge d\bar{z}_{B'}$ , the only addendum containing  $\partial_{\alpha\bar{\beta}}^2\delta$  is

$$r_{A\bar{B}}(z') \otimes \partial_{\alpha\bar{\beta}}^2\delta(z'').$$

As  $\partial\bar{\partial}T = 0$ , we get  $r_{A\bar{B}} = 0$ . Therefore  $\text{supp } T \subseteq \text{Sing } Y$ , and we get the result by induction on the dimension of  $Y$ . q.e.d.

The next result is a vanishing lemma based on the Kodaira-Nakano Vanishing Theorem.

**2.2. Lemma.** *Let  $f: \tilde{M} \rightarrow M$  be the blow-up of  $M$  along a submanifold  $Y$ . Then  $H^0(E, \Omega_E^1(N_{E|\tilde{M}})) = 0$ , where  $N_{E|\tilde{M}}$  is the normal bundle of the exceptional set  $E$ .*

*Proof.* Identify  $f$  locally with the blow-up  $\pi$  as said in (1.1). Take  $y \in U \cap Y = B_m$  and identify the singular fibre  $\pi^{-1}(y)$  with  $\mathbf{P}_{n-1}$ . Call  $E_U := E \cap U$ . Let us recall some easy facts about normal and conormal bundles:

(i) The conormal bundle  $N^*$  is defined by the following exact sequence of vector spaces, for  $x \in \mathbf{P}_{n-1}$ :

$$0 \rightarrow N_{\mathbf{P}_{n-1}|E_U, x}^* \rightarrow T_{E_U, x}'^* \rightarrow T_{\mathbf{P}_{n-1}, x}'^* \rightarrow 0.$$

(ii) Since  $E_U = B_m \times \mathbf{P}_{n-1}$ , the conormal bundle  $N_{\mathbf{P}_{n-1}|E_U}^*$  is trivial.

(iii)  $N_{\mathbf{P}_{n-1}|\tilde{B}_n}^* = [\mathbf{P}_{n-1}]|_{\mathbf{P}_{n-1}} = [-H]$  (notations are the standard ones, see, e.g., [8]).

From (i) we get

$$(2.1) \quad \begin{aligned} 0 \rightarrow \mathcal{O}(N_{\mathbf{P}_{n-1}|E_U}^*) \otimes \mathcal{O}(N_{\mathbf{P}_{n-1}|\tilde{B}_n}) &\rightarrow \Omega_{E_U|\mathbf{P}_{n-1}}^1 \otimes \mathcal{O}(N_{\mathbf{P}_{n-1}|\tilde{B}_n}) \\ &\rightarrow \Omega_{\mathbf{P}_{n-1}}^1 \otimes \mathcal{O}(N_{\mathbf{P}_{n-1}|\tilde{B}_n}) \rightarrow 0 \end{aligned}$$

and from (ii) and (iii) we infer that

$$\mathcal{O}(N_{\mathbf{P}_{n-1}|E_U}^*) \otimes \mathcal{O}(N_{\mathbf{P}_{n-1}|\tilde{B}_n}) = \mathcal{O}(-1).$$

Hence the long exact sequence of cohomology groups of (2.1) starts with

$$\begin{aligned} 0 \rightarrow H^0(\mathbf{P}_{n-1}, \mathcal{O}(-1)) &\rightarrow H^0(\mathbf{P}_{n-1}, \Omega_{E_U|\mathbf{P}_{n-1}}^1 \otimes \mathcal{O}(-1)) \\ &\rightarrow H^0(\mathbf{P}_{n-1}, \Omega^1(-1)) \rightarrow \dots \end{aligned}$$



From the Kodaira-Nakano Vanishing Theorem (and some easy facts about Riemann surfaces for  $n = 2$ ), we get

$$H^0(\mathbf{P}_{n-1}, \Omega^1(-1)) = H^0(\mathbf{P}_{n-1}, \mathcal{O}(-1)) = 0;$$

thus  $H^0(\mathbf{P}_{n-1}, \Omega^1_{E_U}|_{\mathbf{P}_{n-1}} \otimes \mathcal{O}(-1)) = 0$ .

Let  $h \in H^0(E, \Omega^1_{E_U}(N_{E_U|\tilde{U}}))$ . Since

$$N_{E_U|\tilde{U}}|_{\mathbf{P}_{n-1}} = N_{\mathbf{P}_{n-1}|\tilde{E}_n},$$

$h|_{\mathbf{P}_{n-1}}$  is a section of  $\Omega^1_{E_U}|_{\mathbf{P}_{n-1}} \otimes \mathcal{O}(N_{\mathbf{P}_{n-1}|\tilde{E}_n}) = \Omega^1_{E_U}|_{\mathbf{P}_{n-1}} \otimes \mathcal{O}(-1)$  (by iii)). Thus  $h|_{\mathbf{P}_{n-1}} = 0$ ; i.e.  $h$ , restricted to a generic fibre  $\pi^{-1}(y)$ , is zero. This achieves the proof. q.e.d.

Now we are ready to prove the following result.

**2.3. Theorem.** *Let  $f: \tilde{M} \rightarrow M$  be the blow-up of  $M$  along a submanifold  $Y$ . If  $T$  is a real  $\partial\bar{\partial}$ -closed current on  $\tilde{M}$  of order zero and degree  $(1, 1)$  whose support is contained in the exceptional set  $E$ , then there exists a pluriharmonic function  $h: Y \rightarrow \mathbf{R}$  such that*

$$T = (h \circ f)[E].$$

Moreover, if  $T$  is a (weak) limit of currents which are components of boundaries, then  $T = 0$ .

*Proof.* Let us fix a coordinate neighborhood  $(V, v_1, \dots, v_N) = (V, (v', v_N))$  in  $\tilde{M}$  such that  $V \cap E = \{v_N = 0\}$ . In  $V$ ,  $T$  has the following expression:

$$T = \frac{i}{2} \sum_{\alpha, \beta=1}^N t_{\alpha\bar{\beta}}(v) dv_\alpha \wedge d\bar{v}_\beta.$$

As  $\text{supp } T \subseteq E$ ,

$$t_{\alpha\bar{\beta}}(v) = r_{\alpha\bar{\beta}}(v') \otimes \delta(v_N)$$

where  $r_{\alpha\bar{\beta}}$  is a measure and  $r_{\beta\alpha} = \bar{r}_{\alpha\bar{\beta}}$ .

Fix  $\alpha, \beta < N$  and compute  $i\partial\bar{\partial}T$ . The coefficient of  $dv_\alpha \wedge dv_N \wedge d\bar{v}_\beta \wedge d\bar{v}_N$ , which has to vanish, is given by

$$\begin{aligned} & -r_{\alpha\bar{\beta}}(v') \otimes \partial_{N\bar{N}}^2 \delta(v_N) + \partial_\alpha r_{N\bar{\beta}}(v') \otimes \partial_{N\bar{N}} \delta(v_N) + \partial_{\bar{\beta}} r_{\alpha\bar{N}}(v') \otimes \partial_N \delta(v_N) \\ & - \partial_{\alpha\bar{\beta}}^2 r_{N\bar{N}}(v') \otimes \delta(v_N). \end{aligned}$$

Hence we conclude that

$$(2.2) \quad \begin{cases} r_{\alpha\bar{\beta}} = 0 & \text{for } 1 \leq \alpha, \beta < N, \\ r_{\alpha\bar{N}} & \text{is holomorphic for } 1 \leq \alpha < N, \\ r_{N\bar{N}} & \text{is pluriharmonic.} \end{cases}$$

Let us now check what happens in another chart. Choose another coordinate neighborhood  $(W, w_1, \dots, w_N) = (W, w', w_N)$  with  $W \cap V \neq \emptyset$  and  $W \cap E = \{w_N = 0\}$ . Assume

$$T = \frac{i}{2} \sum_{\lambda, \mu=1}^N s_{\lambda\bar{\mu}}(w') \otimes \delta(w_N) dw_\lambda \wedge d\bar{w}_\mu \quad \text{in } W,$$

by (2.2) and similar results for  $\{s_{\lambda\bar{\mu}}\}$ , and using the fact that  $\partial v_N / \partial w_\lambda = \partial w_N / \partial v_\alpha = 0$  on  $E$  for  $\alpha, \lambda < N$ , we obtain the following relations:

$$r_{\alpha\bar{N}}(v') = \sum_{\lambda=1}^{N-1} s_{\lambda\bar{N}}(w'(v')) \frac{\partial v_N}{\partial w_N} \frac{\partial w_\lambda}{\partial v_\alpha} \quad \text{if } \alpha < N,$$

and

$$\begin{aligned} r_{N\bar{N}}(v') &= \sum_{\lambda=1}^{N-1} s_{\lambda\bar{N}}(w'(v')) \frac{\partial v_N}{\partial w_N} \frac{\partial w_\lambda}{\partial v_N} \\ &+ \sum_{\mu=1}^{N-1} s_{\mu\bar{N}}(w'(v')) \frac{\partial v_N}{\partial w_N} \frac{\partial w_\mu}{\partial v_N} + s_{N\bar{N}}(w'(v')). \end{aligned}$$

But now it is a matter of course that, if we cover  $\tilde{M}$  by charts of type  $(V, v', v_N)$ , then  $\{r_{1\bar{N}}, \dots, r_{n-1\bar{N}}\}$  is nothing but a global section of  $\Omega_E^1(N_E|_{\tilde{M}})$  on  $E$ : Indeed, the cocycles of  $N_E|_{\tilde{M}}$  are given by  $\partial v_N / \partial w_N$ . By Lemma 2.2 we get  $r_{\alpha\bar{N}} = s_{\lambda\bar{N}} = 0$ , and  $r_{N\bar{N}} = s_{N\bar{N}} =: h$  is a pluriharmonic map on  $\tilde{M}$ . Since the fibers of  $f$  are compact,  $h$  depends only on the coordinates of  $Y$ .

To get the second part of the statement, by (1.3) we need only to prove that if  $h: Y \rightarrow \mathbf{R}$  is not identically zero, then there exists a closed form  $\Theta \in \mathcal{D}^{N-1, N-1}(\tilde{M})_{\mathbf{R}}$  such that

$$T(\Theta) = \int_E (h \circ f) \Theta \neq 0.$$

Choose  $y \in Y$  such that  $h(y) \neq 0$  (suppose  $h(y) > 0$ ); choose an open neighborhood  $U$  of  $y$ , biholomorphic to  $B_m \times B_n$  and such that  $U \cap Y \cong B_m \times \{0\}$  and  $h > 0$  in  $U \cap Y$ ; then identify  $f|_{f^{-1}(U)}$  with the blow-up  $\pi$ . Take real functions  $u$  and  $v$  as follows:

$$\begin{aligned} u &\in \mathcal{C}_0^\infty(B_m), \quad u \neq 0, \quad u(t) \geq 0, \\ v &\in \mathcal{C}_0^\infty(B_n), \quad v(z) = 1 \text{ near the origin.} \end{aligned}$$

Then assume

$$\psi(z) = \frac{i}{2\pi} \partial \bar{\partial} ((1 - v(z)) \log \|z\|^2),$$

$$\Theta = \left( \frac{i}{2\pi} \partial \bar{\partial} \log \|\xi\|^2 - \pi^* \psi \right)^{n-1} \wedge u(t) \left( \frac{i}{2} \partial \bar{\partial} \|t\|^2 \right)^m.$$

It is straightforward to verify that  $\Theta$  satisfies what is required. In particular, if  $i: E \rightarrow f^{-1}(U)$  is the inclusion,  $i^* \Theta = (\frac{i}{2\pi} \partial \bar{\partial} \log \|\xi\|^2)^{n-1} \wedge u(t) (\frac{i}{2} \partial \bar{\partial} \|t\|^2)^m$ ; hence

$$(h \circ f)[E](\Theta) = \int_E (h \circ f \circ i)(i^* \Theta) = c \int_{B_m} h(t) u(t) \left( \frac{i}{2} \partial \bar{\partial} \|t\|^2 \right)^m > 0.$$

**2.4. Remark.** An example of a  $\partial \bar{\partial}$ -closed current  $T$  of order zero which is not locally flat is given here. Let us use the same notation of the previous theorem, and define

$$T = \frac{i}{2} c \delta(v_N) (dv_1 \wedge d\bar{v}_N + dv_N \wedge d\bar{v}_1), \quad c \in \mathbf{R}.$$

$T$  is a real  $\partial \bar{\partial}$ -closed current of degree  $(1, 1)$  and order zero, and  $\text{supp } T \subseteq \{v_N = 0\}$ . By the Support Theorem (see [10, Theorem 1.7]), if  $T$  were locally flat, there would exist a pluriharmonic function  $h: E \rightarrow \mathbf{R}$  such that  $T = h[E]$ , but this is not the case, if  $c \neq 0$ .

**2.5. Proposition.** Let  $f: \widetilde{M} \rightarrow M$  be a proper modification which is obtained as a finite sequence of blow-ups with smooth centers. Call  $\{E_\alpha\}$  the set of the irreducible components of the exceptional set  $E$ . Then:

(i) Every real  $\partial \bar{\partial}$ -closed current  $T$  of order zero and degree  $(1, 1)$  on  $\widetilde{M}$  supported in  $E$  is of the form  $\sum_\alpha u_\alpha [E_\alpha]$ , where  $u_\alpha$  is a pluriharmonic function on  $E_\alpha$ .

(ii) Moreover,  $T$  is a (weak) limit of currents which are components of boundaries if and only if every  $u_\alpha$  vanishes.

*Proof.* By our assumption there is a finite sequence

$$f_j: V_{j+1} \rightarrow V_j, \quad 0 \leq j < r,$$

of blow-ups with smooth centers  $Y_j \subseteq V_j$  and exceptional sets  $E'_{j+1} \subseteq V_{j+1}$  such that  $V_0 = M$ ,  $V_r = \widetilde{M}$ ,  $f = f_0 \circ \dots \circ f_{r-1}$ . By Theorem 2.3 we get

$$(f_1 \circ \dots \circ f_{r-1})_* T = (u_1 \circ f_0)[E'_1] \quad \text{with } u_1: Y_0 \rightarrow \mathbf{R} \text{ pluriharmonic.}$$

Let  $\widetilde{E}'_1$  be the strict transform of  $E'_1$  under  $f_1$ ;  $(u_1 \circ f_0 \circ f_1)[\widetilde{E}'_1]$  is  $\partial \bar{\partial}$ -closed. Therefore we can apply Theorem 2.3 again to obtain

$$(2.3) \quad (f_2 \circ \dots \circ f_{r-1})_* T - (u_1 \circ f_0 \circ f_1)[\widetilde{E}'_1] = (u_2 \circ f_1)[E'_2]$$

with  $u_2: Y_1 \rightarrow \mathbf{R}$  pluriharmonic. Eventually we get

$$T = \sum_{j=1}^r (u_j \circ f_{j-1} \circ \cdots \circ f_{r-1})[\tilde{E}'_j],$$

where  $\tilde{E}'_j$  is the strict transform of  $E'_j$  via  $f_{j-1} \circ \cdots \circ f_{r-1}$ ,  $1 \leq j < r$ ,  $\tilde{E}'_r = E'_r$  and  $u_j: Y_{j-1} \rightarrow \mathbf{R}$  is pluriharmonic.

(ii) Suppose  $T$  is a limit of components of boundaries. Then

$$(f_1 \circ \cdots \circ f_{r-1})_* T = (u_1 \circ f_0)[E'_1]$$

is limit of components of boundaries too so that by Theorem 2.3  $u_1 = 0$ . From (2.3) and Theorem 2.3 we infer  $u_2 = 0$ , and so on. q.e.d.

Let us consider now a generic proper modification  $f: \tilde{M} \rightarrow M$ . The following lemma is essentially contained in [13] to which we refer step by step.

**2.6. Lemma.** *Let  $f: \tilde{M} \rightarrow M$  be a proper modification; for every  $x \in M$  there exist an open neighborhood  $V$  of  $x$  in  $M$ , a complex manifold  $Z$  and holomorphic maps  $g: Z \rightarrow \tilde{M}$ ,  $h: Z \rightarrow V$  such that  $h = f \circ g$ . Moreover  $g: Z \rightarrow f^{-1}(V)$  is a blow-up, and  $h: Z \rightarrow V$  is obtained as a finite sequence of blow-ups with smooth centers.*

*Proof.* Locally,  $f$  is dominated by a blow-up; that is, [13, Lemma 8, p. 321] for every  $x \in M$  there exist an open neighborhood  $V$  of  $x$  in  $M$  and a complex subspace  $(D, \tilde{\mathcal{O}}_D)$  of  $V$  such that, if  $h': V' \rightarrow V$  is the blow-up with center  $(D, \tilde{\mathcal{O}}_D)$ , then there exists a holomorphic map  $g': V' \rightarrow \tilde{M}$  with  $h' = f \circ g'$ . Let  $\mathcal{F}$  be the coherent ideal sheaf in  $\mathcal{O}_V$  which defines the complex space  $(D, \tilde{\mathcal{O}}_D)$ ; by applying Lemma 7 [13, p. 320] to  $V$  and  $\mathcal{F}$  we get a suitable finite sequence

$$h_j: V_{j+1} \rightarrow V_j, \quad 0 \leq j < r,$$

of blow-ups with smooth centers such that  $Z := V_r$  is smooth, and if  $h := h_0 \circ \cdots \circ h_{r-1}$ ,  $h^{-1}(\mathcal{F})$  is invertible (see the remark after Lemma 7 in [13]). Shrinking  $V$  we can also suppose  $V_0 = V$ . Therefore, by means of the universal property of blow-ups [13, Definition 1, p. 315], we get a holomorphic map  $g'': Z \rightarrow V'$  such that  $h = h' \circ g''$ . If  $g := g' \circ g'': Z \rightarrow \tilde{M}$ , then  $h = f \circ g$  and  $g: Z \rightarrow f^{-1}(V)$  becomes a blow-up since  $h: Z \rightarrow V$  is obtained as a finite sequence of blow-ups and  $f: f^{-1}(V) \rightarrow V$  is a proper modification (see Corollary 1, p. 320 and Lemma 4, p. 318 of [13]). q.e.d.

We can prove now the last result of this section.

**2.7. Proposition.** *Let  $f: \widetilde{M} \rightarrow M$  be a proper modification and let  $\{E_\alpha\}$  be the set of irreducible components of the exceptional set  $E$ . If  $T = \sum_\alpha c_\alpha [E_\alpha]$ ,  $c_\alpha \in \mathbf{R}$ , and for every  $x \in M$  there exists an open neighborhood  $V$  of  $x$  such that*

$$(2.4) \quad T|_{f^{-1}(V)} \text{ is a (weak) limit of currents which are components of boundaries,}$$

then  $c_\alpha = 0 \forall \alpha$ .

*Proof.* Fix  $\alpha^\circ$  and choose  $x \in f(E_{\alpha^\circ})$ . For a suitable open neighborhood  $V$  of  $x$  in  $M$  we get holomorphic maps  $g: Z \rightarrow f^{-1}(V)$  and  $h: Z \rightarrow V$  as in the previous lemma. Now  $T|_{f^{-1}(V)} = \sum_{\alpha,j} c_\alpha [E'_{\alpha,j}]$  where  $\{E'_{\alpha,j}\}$  is the set of connected components of  $E_\alpha \cap f^{-1}(V)$ . Let  $\{F_\beta\}$  be the set of irreducible components of the exceptional set of  $g: Z \rightarrow f^{-1}(V)$ , and denote the strict transform of  $E'_{\alpha,j}$  under  $g$  by  $\widetilde{E}'_{\alpha,j}$ . Thus  $\{F_\beta\} \cup \{\widetilde{E}'_{\alpha,j}\}$  is the set of irreducible components of the exceptional set of  $h: Z \rightarrow V$ , and therefore the total transform  $\widehat{T}$  of  $\sum_{\alpha,j} c_\alpha [E'_{\alpha,j}]$  under  $g$  is of the form  $\widehat{T} = \sum_{\alpha,j} c_\alpha [\widetilde{E}'_{\alpha,j}] + \sum_\beta c'_\beta [F_\beta]$ . By Proposition 2.5 we need only to prove that  $\widehat{T}|_{h^{-1}(V)}$  satisfies (2.4).

Let  $\varphi \in \mathcal{E}^{1,1}(f^{-1}(V))_{\mathbf{R}}$  be a representative of the fundamental class of  $\sum_{\alpha,j} c_\alpha [E'_{\alpha,j}]$  in  $H^2(f^{-1}(V), \mathbf{R})$ ; i.e.,  $\varphi = \sum_{\alpha,j} c_\alpha [E'_{\alpha,j}] + dQ$  for a suitable current  $Q$  in  $f^{-1}(V)$ . Then  $g^*\varphi$  represents the fundamental class of the total transform  $\widehat{T}$ ; i.e.,

$$(2.5) \quad g^*\varphi = \widehat{T} + dQ'$$

for a suitable current  $Q'$  in  $Z$ .

The hypothesis (2.4) provides a sequence  $\{R_\mu\}$  of  $(1, 0)$ -currents in  $f^{-1}(V)$  such that

$$\sum_{\alpha,j} c_\alpha [E'_{\alpha,j}] = \lim_\mu (\bar{\partial}R_\mu + \partial\bar{R}_\mu) \quad (\text{weakly}).$$

By smoothing  $R_\mu$  and  $Q$  we get

$$\varphi = \lim_\mu (\bar{\partial}\rho_\mu + \partial\bar{\rho}_\mu) \quad (\text{weakly})$$

where  $\rho_\mu$  are smooth  $(1, 0)$ -forms in  $f^{-1}(V)$ .

Let  $S$  be a closed current of degree  $(N-1, N-1)$  with compact support in  $f^{-1}(V)$  and let  $\psi \in \mathcal{D}^{N-1, N-1}(f^{-1}(V))_{\mathbf{R}}$  such that  $S = \psi + i\partial\bar{\partial}u$

for a suitable current  $u$  with compact support in  $f^{-1}(V)$ . Now,

$$S(\varphi) = \int \varphi \wedge \psi + i\partial\bar{\partial}u(\varphi) = \lim_{\mu} \int (\bar{\partial}\rho_{\mu} + \partial\bar{\rho}_{\mu}) \wedge \psi + i\bar{\partial}u(\partial\varphi) = 0.$$

Finally, consider the operator

$$(\partial \oplus \bar{\partial}): (\mathcal{E}^{0,1}(f^{-1}(V)) \oplus \mathcal{E}^{1,0}(f^{-1}(V)))_{\mathbf{R}} \rightarrow \mathcal{E}^{1,1}(f^{-1}(V))_{\mathbf{R}}$$

and its dual

$$d: (\mathcal{E}^{1,1}(f^{-1}(V))_{\mathbf{R}})' \rightarrow (\mathcal{E}^{0,1}(f^{-1}(V)) \oplus \mathcal{E}^{1,0}(f^{-1}(V)))_{\mathbf{R}}';$$

we have just proved that  $\varphi \in (\text{Ker } d)^{\perp} = \overline{\text{Im}(\partial \oplus \bar{\partial})}$ . Thus  $\varphi$  is limit of components of boundaries in the strong sense, and the same holds for  $g^*\varphi$ ; so, by (2.5), the same holds for  $\hat{T}$  in a weak sense.

### 3. Positive $\partial\bar{\partial}$ -closed currents have a pullback to $\tilde{M}$

In this section we start with the following data:  $f: \tilde{M} \rightarrow M$  is a proper modification and  $T$  is a positive  $\partial\bar{\partial}$ -closed current on  $M$  of degree  $(1, 1)$ , and we try to find a “nice pullback,” say  $\tilde{T}$ , of  $T$  to  $\tilde{M}$ . If  $E$  is the exceptional set of  $f$  and  $Y := f(E)$ , then  $\tilde{M} - E \xrightarrow{f} M - Y$  is a biholomorphism. Therefore such a pullback must extend  $((f|_{\tilde{M}-E})^{-1})_*(T|_{M-Y})$  from  $\tilde{M} - E$  to the whole of  $\tilde{M}$ . What we are looking for is a positive  $\partial\bar{\partial}$ -closed extension  $\tilde{T}$  on  $\tilde{M}$ , which also satisfies

$$(3.1) \quad \forall x \in M, \text{ there exists an open neighborhood } W \text{ of } x \text{ such that } \tilde{T}|_{f^{-1}(W)} \text{ is a (weak) limit of currents which are components of boundaries.}$$

But  $((f|_{\tilde{M}-E})^{-1})_*(T|_{M-Y})$  has an extension of order zero to  $\tilde{M}$  if and only if it has locally finite mass across  $E$  (see [15, p. 10]); i.e.,  $\forall x \in E$ , there is a neighborhood  $V$  of  $x$  in  $\tilde{M}$  such that

$$(3.2) \quad \int_{V-E} ((f|_{\tilde{M}-E})^{-1})_*(T|_{M-Y}) \wedge \theta^{N-1} < \infty,$$

where  $\theta$  is a smooth strictly positive  $(1, 1)$ -form on  $\tilde{M}$ .

Since (3.2) is a local statement, we shall carry on the computations in coordinates, starting with the case of a blow-up with smooth center. After having proved that  $((f|_{\tilde{M}-E})^{-1})_*(T|_{M-Y})$  admits extensions of order zero, we construct an extension  $\tilde{T}$  which satisfies the previous demands (see Theorem 3.9) and such that, if condition (B) holds, its class  $[\tilde{T}]$  in

the Aeppli group  $V^{1,1}(\widetilde{M})_{\mathbb{R}}$  coincides with  $f^*([T])$  (Proposition 3.10). These properties are not enjoyed, in general, by the simple extension  $\widetilde{T}^\circ$ , as an example shows.

To prove the following result, we shall follow [19] (page 129 and ff. for the case  $k = 1$ ); nevertheless, since Siu works with  $d$ -closed currents, we shall give here a sufficiently detailed proof in order to check that his arguments also work in the  $\partial\bar{\partial}$ -closed case and for the sake of completeness.

**3.1. Proposition.** *Let  $\pi$  be the blow-up of  $U := B_m \times B_n$  with center  $Y = B_m \times \{0\}$ , and let  $\tilde{\omega}$  be the Kähler form for  $B_m \times B_n$  defined in (1.1). Suppose  $\{T_\varepsilon\}$  is a family of  $\partial\bar{\partial}$ -closed smooth positive  $(1, 1)$ -forms on  $U$ , such that there is a current  $T$  on  $U$ ,  $T = \lim_\varepsilon T_\varepsilon$  (weakly). Then  $\forall t^\circ \in B_m$ , there exists a neighborhood  $V$  of  $(t^\circ, 0)$  in  $U$  such that*

$$(3.3) \quad \sup_\varepsilon \int_{\pi^{-1}(V)} \pi^* T_\varepsilon \wedge \tilde{\omega}^{N-1} < \infty.$$

*Proof.* Choose a unitary linear coordinates system  $w = w(t, z) = (w_1, \dots, w_N)$  of  $\mathbb{C}^N$  such that  $(w_I, z) := (w_{i_1}, \dots, w_{i_m}, z_1, \dots, z_n)$  form a coordinates system of  $\mathbb{C}^N$  for every  $I = (i_1, \dots, i_m)$  with  $1 \leq i_1 < \dots < i_m \leq N$ . Look at the  $(1, 1)$ -form  $(\frac{i}{2\pi} \partial\bar{\partial} \log \|z\|^2)$ . Its matrix is positive semidefinite; more precisely, at  $z \neq 0$ , it has 0 as simple eigenvalue, with eigendirection  $z$ , and  $1/\pi \|z\|^2$  as eigenvalue of multiplicity  $(n - 1)$ , with eigenspace  $(z)^\perp$ . Hence  $(\frac{i}{2\pi} \partial\bar{\partial} \log \|z\|^2)^h = 0$  if  $h \geq n$ ; this implies that there exists a constant  $c > 0$  such that

$$\begin{aligned} (\pi_* \tilde{\omega})^{m+n-1} &\leq c \sum_{k=m}^{m+n-1} \binom{m+n-1}{k} \sum_I \left( \frac{i}{2\pi} \partial\bar{\partial} \log \|z\|^2 \right)^{m+n-1-k} \\ &\quad \wedge \left( \frac{i}{2} \partial\bar{\partial} \|z\|^2 \right)^{k-m} \wedge \left( \frac{i}{2} \partial\bar{\partial} \|w_I\|^2 \right)^m. \end{aligned}$$

Let  $t^\circ \in B_m$  and let  $\rho_I : \mathbb{C}^N \rightarrow \mathbb{C}^n$  be defined by  $\rho_I(t, z) = w_I(t, z)$ . There exist an open ball  $A_I$  with center  $w_I(t^\circ, 0)$  in  $\mathbb{C}^m$  and  $r_I > 0$  such that

$$X_I := \rho_I^{-1}(A_I) \cap (\mathbb{C}^m \times B_n(r_I)) \Subset U.$$

Thus, if we take  $V := \bigcap_I X_I$ , to check (3.3) we have only to prove that  $\forall I, \forall k, m \leq k \leq m + n - 1$ ,

$$(3.4) \quad \begin{aligned} \sup_\varepsilon \int_{X_I} T_\varepsilon \wedge \left( \frac{i}{2\pi} \partial\bar{\partial} \log \|z\|^2 \right)^{m+n-1-k} &\quad \wedge \left( \frac{i}{2} \partial\bar{\partial} \|z\|^2 \right)^{k-m} \\ &\quad \wedge \left( \frac{i}{2} \partial\bar{\partial} \|w_I\|^2 \right)^m < \infty. \end{aligned}$$

The form which is integrated in (3.4) is smaller than or equal to

$$\frac{1}{(\pi \|z\|^2)^{N-1-k}} T_\varepsilon \wedge \left( \frac{i}{2} \partial \bar{\partial} \|z\|^2 \right)^{n-1} \wedge \left( \frac{i}{2} \partial \bar{\partial} \|w_I\|^2 \right)^m,$$

which is in  $L_{\text{loc}}^1(U)$ ; this implies that in (3.4) we can ignore the singularity of  $\partial \bar{\partial} \log \|z\|^2$ . Since  $i \partial \bar{\partial} T_\varepsilon = 0$ , there exist  $(1, 0)$ -forms  $S_\varepsilon$  on  $U = B_m \times B_n$  such that  $T_\varepsilon = \bar{\partial} S_\varepsilon + \partial \bar{S}_\varepsilon$ . Thus denoting the (topological) boundary of  $X_I$  by  $bX_I$

$$\begin{aligned} & \int_{X_I} T_\varepsilon \wedge \left( \frac{i}{2\pi} \partial \bar{\partial} \log \|z\|^2 \right)^{m+n-1-k} \wedge \left( \frac{i}{2} \partial \bar{\partial} \|z\|^2 \right)^{k-m} \wedge \left( \frac{i}{2} \partial \bar{\partial} \|w_I\|^2 \right)^m \\ &= \int_{bX_I} (S_\varepsilon + \bar{S}_\varepsilon) \wedge \left( \frac{i}{2\pi} \partial \bar{\partial} \log \|z\|^2 \right)^{m+n-1-k} \\ & \quad \wedge \left( \frac{i}{2} \partial \bar{\partial} \|z\|^2 \right)^{k-m} \wedge \left( \frac{i}{2} \partial \bar{\partial} \|w_I\|^2 \right)^m \\ &= \frac{1}{(\pi r_I^2)^{m+n-1-k}} \int_{bX_I} (S_\varepsilon + \bar{S}_\varepsilon) \wedge \left( \frac{i}{2} \partial \bar{\partial} \|z\|^2 \right)^{n-1} \wedge \left( \frac{i}{2} \partial \bar{\partial} \|w_I\|^2 \right)^m \\ &= \frac{1}{(\pi r_I^2)^{m+n-1-k}} \int_{X_I} T_\varepsilon \wedge \left( \frac{i}{2} \partial \bar{\partial} \|z\|^2 \right)^{n-1} \wedge \left( \frac{i}{2} \partial \bar{\partial} \|w_I\|^2 \right)^m, \end{aligned}$$

where the reason for the second equality is the following:

$$bX_I := [\rho_I^{-1}(bA_I) \cap (\mathbf{C}^m \times B_n(r_I))] \cup [\rho_I^{-1}(A_I) \cap (\mathbf{C}^m \times bB_n(r_I))] = Y_1 \cup Y_2;$$

integration on  $Y_1$  gives no contribution because  $(\frac{i}{2} \partial \bar{\partial} \|w_I\|^2)^m$  is a  $2m$ -form on the manifold  $bA_I$  of real dimension  $2m-1$ . On the other hand, on  $Y_2$  we have

$$\frac{i}{2\pi} \partial \bar{\partial} \log \|z\|^2 = \frac{i}{2\pi r_I^2} \partial \bar{\partial} \|z\|^2.$$

Now we need the following result [19, p. 66].

**3.2. Lemma.** *Suppose  $G_1 \Subset G_2 \Subset U$  are relatively compact open subsets of  $\mathbf{C}^N$ ,  $\varphi$  is a product of  $(N-k)$  smooth positive  $(1, 1)$ -forms and  $\{T_\varepsilon\}$  is a sequence of positive currents on  $U$  of degree  $(k, k)$  converging (weakly) to a current  $T$  on  $U$ . Then*

$$\limsup_\varepsilon \int_{G_1} T_\varepsilon \wedge \varphi \leq \int_{G_2} T \wedge \varphi \quad \text{and} \quad \int_{G_1} T \wedge \varphi \leq \liminf_\varepsilon \int_{G_2} T_\varepsilon \wedge \varphi.$$



If we choose  $G$  such that  $X_I \in G \Subset U$ , by virtue of (3.5) and Lemma 3.2 we get

$$\begin{aligned} & \limsup_\varepsilon \int_{X_I} T_\varepsilon \wedge \left( \frac{i}{2\pi} \partial \bar{\partial} \log \|z\|^2 \right)^{m+n-1-k} \\ & \quad \wedge \left( \frac{i}{2} \partial \bar{\partial} \|z\|^2 \right)^{k-m} \wedge \left( \frac{i}{2} \partial \bar{\partial} \|w_I\|^2 \right)^m \\ & \leq \frac{1}{(\pi r_I^2)^{m+n-1-k}} \int_G T \wedge \left( \frac{i}{2} \partial \bar{\partial} \|z\|^2 \right)^{n-1} \wedge \left( \frac{i}{2} \partial \bar{\partial} \|w_I\|^2 \right)^m < \infty. \end{aligned}$$

Thus also

$$\begin{aligned} \sup_\varepsilon \int_{X_I} T_\varepsilon \wedge \left( \frac{i}{2\pi} \partial \bar{\partial} \log \|z\|^2 \right)^{m+n-1-k} & \wedge \left( \frac{i}{2} \partial \bar{\partial} \|z\|^2 \right)^{k-m} \\ & \wedge \left( \frac{i}{2} \partial \bar{\partial} \|w_I\|^2 \right)^m < \infty. \quad \text{q.e.d.} \end{aligned}$$

We would like to mention the following easy consequence of the above lemma, which is used in what follows:

(3.6) If  $\{T_\varepsilon\}$  and  $\varphi$  are as in Lemma 3.2, and  $L$  is a Borel set,  $L \Subset U$ , then

$$\lim_\varepsilon \int_L T_\varepsilon \wedge \varphi = \int_L T \wedge \varphi \text{ if } \|T\|(bL) = 0.$$

**3.3. Proposition.** *Let  $f: \widetilde{M} \rightarrow M$  be the blow-up of  $M$  along a submanifold  $Y$ . If  $T$  is a positive  $\partial \bar{\partial}$ -closed current on  $M$  of degree  $(1, 1)$ , then the current  $((f|_{\widetilde{M}-E})^{-1})_*(T|_{M-Y})$  has locally finite mass across  $E$ , and hence extends to a current of order zero.*

*Proof.* Identify  $f$  locally with the blow-up  $\pi$  and let  $y = (t^\circ, 0) \in Y$ . By smoothing  $T$  by convolutions in a suitable open neighborhood  $U$  of  $y$  in  $M$ , we get a family  $\{T_\varepsilon\}$  as in Proposition 3.1. Choose  $r > 0$  and a sequence  $\{r_j\}$  of positive real numbers such that  $r_j \downarrow 0$ ,  $V_0 := B_m(t^\circ, r) \times B_n(r_0) \Subset U$  and, for every  $j \geq 0$ ,

(3.7)  $\|T\|(bV_j) = 0$ , where  $V_j := B_m(t^\circ, r) \times B_n(r_j)$ .

Since

$$\lim_j \chi_{\pi^{-1}(V_0 - V_j)} = \chi_{\pi^{-1}(V_0 - Y)},$$

where, as usual,  $\chi_L$  is the characteristic function of  $L$ , we infer that

$$\begin{aligned} \lim_j \int_{\pi^{-1}(V_0 - V_j)} ((\pi|_{\tilde{U}-E})^{-1})_* (T|_{U-Y}) \wedge \tilde{\omega}^{N-1} \\ = \int_{\pi^{-1}(V_0 - Y)} ((\pi|_{\tilde{U}-E})^{-1})_* (T|_{U-Y}) \wedge \tilde{\omega}^{N-1}, \end{aligned}$$

where  $\tilde{U} := f^{-1}(U)$ ; but

$$\int_{\pi^{-1}(V_0 - V_j)} ((\pi|_{\tilde{U}-E})^{-1})_* (T|_{U-Y}) \wedge \tilde{\omega}^{N-1} = \int_{V_0 - V_j} T \wedge \pi_* \tilde{\omega}^{N-1}.$$

By (3.7) and (3.6), we have

$$\begin{aligned} \int_{V_0 - V_j} T \wedge \pi_* \tilde{\omega}^{N-1} &= \lim_\varepsilon \int_{V_0 - V_j} T_\varepsilon \wedge \pi_* \tilde{\omega}^{N-1} \\ &\leq \sup_\varepsilon \int_{V_0} T_\varepsilon \wedge \pi_* \tilde{\omega}^{N-1} < \infty. \end{aligned}$$

Thus

$$\int_{\pi^{-1}(V_0 - Y)} ((\pi|_{\tilde{U}-E})^{-1})_* (T|_{U-Y}) \wedge \tilde{\omega}^{N-1} < \infty.$$

**Remark.** One of the possible extensions of order zero is the “simple extension” [15], which is defined as follows:

$$\tilde{T}^\circ(\varphi) = \int_{\tilde{M}-E} ((f|_{\tilde{M}-E})^{-1})_* (T|_{M-Y}) \wedge \varphi$$

for every  $\varphi \in \mathcal{D}^{N-1, N-1}(\tilde{M})_{\mathbf{R}}$ .  $\tilde{T}^\circ$  is called also “extension by zero” since  $\|\tilde{T}^\circ\|(E) = 0$ . Nevertheless we are interested in an extension  $\tilde{T}$  which is also  $\partial\bar{\partial}$ -closed and satisfies (3.1), therefore we go on otherwise.

First we recall a lemma [19, p. 69].

**3.4. Lemma.** *Suppose  $\Omega$  is an open subset of  $\mathbf{C}^N$  and  $\theta$  a strictly positive  $(1, 1)$ -form on  $\Omega$ . Suppose  $\{T_\lambda\}$  is a sequence of smooth positive  $(k, k)$ -forms on  $\Omega$  which satisfy*

$$\sup_\lambda \int_K T_\lambda \wedge \theta^{N-k} < \infty$$

for every compact  $K$  of  $\Omega$ . Then there exists a subsequence  $\{T_{\lambda_\mu}\}$  of  $\{T_\lambda\}$  which converges (weakly) on  $\Omega$ .

Using this result, we are able to complete Proposition 3.1 as follows.

**3.5. Corollary.** *In the hypotheses of Proposition 3.1, there exists a sequence  $\{\varepsilon_\mu\}$ ,  $\varepsilon_\mu \rightarrow 0$ , such that  $\pi^* T_{\varepsilon_\mu}$  converges (weakly) on  $\tilde{U}$  to a*

current  $\tilde{T}_U$ . This current does depend not on the sequence  $\varepsilon_\mu$  but only on  $T$ .

*Proof.* We can take coordinates open sets  $\Omega_j$  such that  $\tilde{U} = \bigcup_{j=1}^n \Omega_j$ . By Proposition 3.1,  $\Omega_j, \pi^* T_\varepsilon|_{\Omega_j}$ , and  $\theta = \hat{\omega}|_{\Omega_j}$  satisfy the hypothesis of Lemma 3.4. Therefore, if we consider subsequently  $\Omega_1, \dots, \Omega_n$ , we find a sequence  $\{\varepsilon_\mu\}$  such that  $\pi^* T_{\varepsilon_\mu}$  converges on each  $\Omega_j$ , hence on  $\tilde{U}$  to a current  $\tilde{T}'_U$ . Let  $\{T'_{\varepsilon_\mu}\}$  be another sequence, with the same properties of  $\{T_{\varepsilon_\mu}\}$ , such that

$$\lim_{\mu} \pi^* T'_{\varepsilon_\mu} = \tilde{T}'_U.$$

Since  $U \cong B_m \times B_n$ ,  $T'_{\varepsilon_\mu}$  and  $T_{\varepsilon_\mu}$  are components of boundaries; therefore we can get  $\tilde{T}'_U = \tilde{T}_U$  by applying Theorem 2.3 to  $\tilde{T}'_U - \tilde{T}_U$ .

**3.6. Remark.** Another proof of the second part of the previous corollary is given here, to emphasize the link between  $\|\tilde{T}_U\|(E)$  and a kind of “Lelong number of  $T$  along  $Y$ ”.

*Proof.* Let  $\{T'_{\varepsilon_\mu}\}$  be another sequence, with the same properties of  $\{T_{\varepsilon_\mu}\}$ , such that

$$\lim_{\mu} \pi^* T'_{\varepsilon_\mu} = \tilde{T}'_U.$$

Take an open ball  $A \Subset B_m$  and  $B_n(r) \Subset B_n$  such that

$$(3.8) \quad \|\tilde{T}_U\|(b\pi^{-1}(A \times B_n(r))) = \|\tilde{T}'_U\|(b\pi^{-1}(A \times B_n(r))) = 0.$$

Since on  $bB_n(r)$

$$\frac{i}{2\pi} \partial\bar{\partial} \log \|z\|^2 = \frac{i}{2\pi r^2} \partial\bar{\partial} \|z\|^2,$$

we get as in the proof of Proposition 3.1

$$(3.9) \quad \begin{aligned} & \int_{\pi^{-1}(A \times B_n(r))} \pi^* T_{\varepsilon_\mu} \wedge \left( \frac{i}{2\pi} \partial\bar{\partial} \log \|\xi\|^2 \right)^{n-1} \wedge \left( \frac{i}{2} \partial\bar{\partial} \|t\|^2 \right)^m \\ &= \int_{A \times B_n(r)} T_{\varepsilon_\mu} \wedge \left( \frac{i}{2\pi} \partial\bar{\partial} \log \|z\|^2 \right)^{n-1} \wedge \left( \frac{i}{2} \partial\bar{\partial} \|t\|^2 \right)^m \\ &= \frac{1}{(\pi r^2)^{n-1}} \int_{A \times B_n(r)} T_{\varepsilon_\mu} \wedge \left( \frac{i}{2} \partial\bar{\partial} \|z\|^2 \right)^{n-1} \wedge \left( \frac{i}{2} \partial\bar{\partial} \|t\|^2 \right)^m. \end{aligned}$$

Notice that (3.8) implies  $\|T\|(b(A \times B_n(r))) = 0$ ; by letting  $\mu \rightarrow \infty$  in

(3.9) we conclude that

$$\begin{aligned} & \int_{\pi^{-1}(A \times B_n(r))} \tilde{T}_U \wedge \left( \frac{i}{2\pi} \partial \bar{\partial} \log \|\xi\|^2 \right)^{n-1} \wedge \left( \frac{i}{2} \partial \bar{\partial} \|t\|^2 \right)^m \\ &= \frac{1}{(\pi r^2)^{n-1}} \int_{A \times B_n(r)} T \wedge \left( \frac{i}{2} \partial \bar{\partial} \|z\|^2 \right)^{n-1} \wedge \left( \frac{i}{2} \partial \bar{\partial} \|t\|^2 \right)^m, \end{aligned}$$

and the same holds for  $\tilde{T}'_U$ .

By applying Theorem 2.3 to the current  $\tilde{T}_U - \tilde{T}'_U$  we get

$$\begin{aligned} 0 &= \int_{\pi^{-1}(A \times B_n(r))} (\tilde{T}_U - \tilde{T}'_U) \wedge \left( \frac{i}{2\pi} \partial \bar{\partial} \log \|\xi\|^2 \right)^{n-1} \wedge \left( \frac{i}{2} \partial \bar{\partial} \|t\|^2 \right)^m \\ &= \int_A h \left( \frac{i}{2} \partial \bar{\partial} \|t\|^2 \right)^m, \end{aligned}$$

and since  $A$  is arbitrary,  $h = 0$ ; i.e.,  $\tilde{T}_U = \tilde{T}'_U$ . q.e.d.

We would like also to compare the previous proof with the analogous situation occurring in [19, p. 129]. The author has a closed current  $\tilde{T}_U$ ; hence he gets

$$\chi_{E_U} \tilde{T}_U = c[E_U]$$

by deep results based on Bombieri-Hörmander estimates (see [19, Chapter 5]). Next,

$$\chi_{E_U} \tilde{T}'_U = c'[E_U],$$

for another  $\tilde{T}'_U$  implies  $\tilde{T}_U - \tilde{T}'_U = (c - c')[E_U]$ . But the class of  $[E_U]$  does not vanish in the cohomology ring of  $\tilde{U}$ , while the class of  $\tilde{T}_U - \tilde{T}'_U$  is zero, because  $\tilde{T}_U$  and  $\tilde{T}'_U$  are limits of sequences of boundaries; this implies  $c = c'$ .

**3.7. Remark.** Let  $f: \tilde{M} \rightarrow M$  be the blow-up of  $M$  along a submanifold  $Y$ , and let  $T$  be a positive  $\partial \bar{\partial}$ -closed current on  $M$  of degree  $(1, 1)$ . For every  $x \in M$ , there exists an open neighborhood  $U$  of  $x$  such that, smoothing  $T$  by convolutions, we can apply Corollary 3.5 to get a positive extension  $\tilde{T}_U$  of  $((f|_{\tilde{U}-E})^{-1})_*(T|_{U-Y})$  in  $f^{-1}(U)$ . By construction, such an extension is the limit of currents which are components of boundaries; hence  $\tilde{T}_U$  is  $\partial \bar{\partial}$ -closed. But in general  $\tilde{T}^\circ \neq \tilde{T}_U$ , as the following example shows.

**Example.** Let us take  $Y = \{0\}$  (that is,  $m = 0$  and  $N = n$ ), and let us consider a sequence  $\{T_\varepsilon\}$  as in Proposition 3.1. For  $0 < r_1 < r_2 < 1$ ,

$$\begin{aligned} & \int_{r_1 < \|z\| < r_2} T_\varepsilon \wedge \pi_* \tilde{\omega}^{N-1} \\ &= \sum_{h=0}^{N-1} \binom{N-1}{h} \int_{r_1 < \|z\| < r_2} T_\varepsilon \wedge \left( \frac{i}{2\pi} \partial \bar{\partial} \log \|z\|^2 \right)^h \wedge \left( \frac{i}{2} \partial \bar{\partial} \|z\|^2 \right)^{N-1-h} \\ &= \sum_{h=0}^{N-1} \binom{N-1}{h} \frac{1}{(\pi r_2^2)^h} \int_{\|z\| < r_2} T_\varepsilon \wedge \left( \frac{i}{2} \partial \bar{\partial} \|z\|^2 \right)^{N-1} \\ &\quad - \sum_{h=0}^{N-1} \binom{N-1}{h} \frac{1}{(\pi r_1^2)^h} \int_{\|z\| < r_1} T_\varepsilon \wedge \left( \frac{i}{2} \partial \bar{\partial} \|z\|^2 \right)^{N-1}. \end{aligned}$$

For the second equality, see [20, p. 364, Remark 1] or also our (3.5). For  $r_1 \rightarrow 0$ ,

$$\int_{\|z\| < r_2} T_\varepsilon \wedge \pi_* \tilde{\omega}^{N-1} = \sum_{h=0}^{N-1} \binom{N-1}{h} \frac{1}{(\pi r_2^2)^h} \int_{\|z\| < r_2} T_\varepsilon \wedge \left( \frac{i}{2} \partial \bar{\partial} \|z\|^2 \right)^{N-1}.$$

Now choose the subsequence  $\varepsilon_\mu$  of Corollary 3.5 and  $r_2$  such that  $\|T\|(bB_n(r_2)) = 0$ . For  $\mu \rightarrow \infty$

$$\begin{aligned} \|\tilde{T}_U\|(\pi^{-1}(B_n(r_2))) &= \int_{\pi^{-1}(\|z\| < r_2)} \tilde{T}_U \wedge \tilde{\omega}^{N-1} \\ &= \lim_{\mu} \int_{\pi^{-1}(\|z\| < r_2)} \pi^* T_{\varepsilon_\mu} \wedge \tilde{\omega}^{N-1} = \lim_{\mu} \int_{\|z\| < r_2} T_{\varepsilon_\mu} \wedge \pi_* \tilde{\omega}^{N-1} \\ &= \lim_{\mu} \sum_{h=0}^{N-1} \binom{N-1}{h} \frac{1}{(\pi r_2^2)^h} \int_{\|z\| < r_2} T_{\varepsilon_\mu} \wedge \left( \frac{i}{2} \partial \bar{\partial} \|z\|^2 \right)^{N-1}; \end{aligned}$$

thus

$$(3.10) \quad \|\tilde{T}_U\|(E \cap U) = \lim_{r_2 \rightarrow 0} \|\tilde{T}_U\|(\pi^{-1}(B_n(r_2))) = n(T, 0).$$

Hence, if the Lelong number  $n(T, 0) \neq 0$ ,  $\tilde{T}^\circ \neq \tilde{T}_U$ .

It is perhaps interesting to check the difference between  $\tilde{T}^\circ$  and  $\tilde{T}_U$ . Take  $T = [H]$ , where  $H = \{z_n = 0\}$ ; we get easily that  $\tilde{T}^\circ$  is nothing else than the strict transform of  $H$  under  $\pi$ . Hence  $\tilde{T}^\circ$  is closed, and therefore  $\tilde{T} - \tilde{T}^\circ$  is  $\partial \bar{\partial}$ -closed; obviously

$$\tilde{T} - \tilde{T}^\circ \geq 0 \quad \text{and} \quad \text{supp}(\tilde{T} - \tilde{T}^\circ) \subseteq E = \mathbf{P}_{n-1}.$$

By Theorem 1.5 in [2], there exists a constant  $k \geq 0$  such that  $\tilde{T} - \tilde{T}^\circ = k[E]$ . This implies

$$\|\tilde{T}\|(E) = k \text{vol}(\mathbf{P}_{n-1}) = k,$$

but from (3.10)

$$\|\tilde{T}\|(E) = n([H], 0) = 1.$$

Thus  $\tilde{T} = \tilde{T}^\circ + [E]$ .

This example reflects a more general situation, which is discussed in Proposition 3.10: that is, if  $T$  is a divisor, its total transform is  $\tilde{T}$  and not, in general,  $\tilde{T}^\circ$ .

Let us collect our results in the following theorem.

**3.8. Theorem.** *Let  $f: \tilde{M} \rightarrow M$  be the blow-up of  $M$  along a submanifold  $Y$ . If  $T$  is a positive  $\partial\bar{\partial}$ -closed current on  $M$  of degree  $(1, 1)$ , then  $((f|_{\tilde{M}-E})^{-1})_*(T|_{M-Y})$  can be extended to  $\tilde{M}$ , and there exists an extension  $\tilde{T}$  which is positive and  $\partial\bar{\partial}$ -closed, and satisfies the following condition:*

$$(3.1) \quad \forall x \in M, \text{ there exists an open neighborhood } W \text{ of } x \text{ such that } \tilde{T}|_{f^{-1}(W)} \text{ is a (weak) limit of currents which are components of boundaries.}$$

Now we consider the general case of a proper modification.

**3.9. Theorem.** *Let  $f: \tilde{M} \rightarrow M$  be a proper modification and let  $T$  be a positive  $\partial\bar{\partial}$ -closed current on  $M$  of degree  $(1, 1)$ . Then the following hold:*

(i) *There exists a positive  $\partial\bar{\partial}$ -closed current  $\tilde{T}$  on  $\tilde{M}$  of degree  $(1, 1)$  such that  $f_*\tilde{T} = T$  and (3.1) holds.*

(ii) *If  $M$  is compact, such a current  $\tilde{T}$  is unique.*

*Proof.* (i) Let  $x \in M$  and choose an open neighborhood  $V$  of  $x$  and maps  $g: Z \rightarrow f^{-1}(V)$  and  $h: Z \rightarrow V$  as in Lemma 2.6. Since  $h$  is obtained as a finite sequence of blow-ups with smooth centers, by Theorem 3.8 we get a positive  $\partial\bar{\partial}$ -closed current  $\hat{T}$  on  $Z$  such that  $h_*\hat{T} = T$ . Shrinking  $V$ , we can suppose that it is contained in a coordinate chart and is biholomorphic to an open ball, and that there exists a sequence  $\{T_\epsilon\}$  of components of boundaries,  $T_\epsilon \rightarrow T$  (e.g., by smoothing  $T$  by convolutions). By construction,  $\hat{T} = \lim_\epsilon h^*T_\epsilon$  and it does not depend on  $\{T_\epsilon\}$ . Define

$$\tilde{T}_V := g_*\hat{T} = \lim_\epsilon g_*h^*T_\epsilon = \lim_\epsilon (f|_{f^{-1}(V)})^*T_\epsilon.$$

Since  $\tilde{T}_V$  does not depend on  $\{T_\epsilon\}$  nor on the factorization  $h = f \circ g$ ,  $\tilde{T}|_{f^{-1}(V)} := \tilde{T}_V$  defines the required current.

(ii) Let  $\tilde{T}$  and  $\tilde{T}'$  be currents on  $\tilde{M}$  which satisfy (i). Let  $x \in M$  and  $Z, g, h$  be as above; using (i) for the map  $g$ , we get positive  $\partial\bar{\partial}$ -closed currents  $\hat{T}$  and  $\hat{T}'$  on  $Z$  such that  $g_*\hat{T} = \tilde{T}$  and  $g_*\hat{T}' = \tilde{T}'$  on

$f^{-1}(V)$ ; hence  $h_*\hat{T} = h_*\hat{T}' = T$  on  $V$ . Let  $\{E'_\gamma\}$  be the set of irreducible components of  $E \cap f^{-1}(V)$ , and  $\{F_\beta\}$  be the set of irreducible components of the exceptional set  $F$  of  $g: Z \rightarrow f^{-1}(V)$ , and denote by  $\tilde{E}'_\gamma$  the strict transform of  $E'_\gamma$  under  $g$ . Thus  $\{F_\beta\} \cup \{\tilde{E}'_\gamma\}$  is the set of irreducible components of the exceptional set of  $h: Z \rightarrow V$ , and by Proposition 2.5 we get

$$\hat{T} - \hat{T}' = \sum_\gamma u_\gamma[\tilde{E}'_\gamma] + \sum_\beta u_\beta[F_\beta],$$

where  $u_\gamma$  and  $u_\beta$  are pluriharmonic functions. Thus  $\tilde{T} - \tilde{T}' = g_*(\hat{T} - \hat{T}') = \sum_\gamma u'_\gamma[E'_\gamma]$  on  $f^{-1}(V)$ , where  $u'_\gamma$  is a well-defined pluriharmonic function on  $E'_\gamma - g(\tilde{E}'_\gamma \cap F)$ , locally bounded on  $E'_\gamma$ . Hence  $u'_\gamma$  extends to  $E'_\gamma$  so that we get on  $\tilde{M}$

$$\tilde{T} - \tilde{T}' = \sum_\alpha u_\alpha[E_\alpha],$$

where  $\{E_\alpha\}$  is the set of irreducible components of  $E$ , and  $u_\alpha$  is pluriharmonic on  $E_\alpha$ . But each  $E_\alpha$  is compact, so each  $u_\alpha$  is constant. The thesis follows from Proposition 2.7. *q.e.d.*

Let us consider now the class in  $V^{1,1}(\tilde{M})_{\mathbb{R}}$  of the current  $\tilde{T}$  given by the previous theorem.

**3.10. Proposition.** *Let  $M, \tilde{M}$  be complex manifolds which satisfy condition (B), and let  $f: \tilde{M} \rightarrow M$  be a proper modification. Let  $T$  be a positive  $\partial\bar{\partial}$ -closed current on  $M$  of degree  $(1, 1)$ , and  $\tilde{T}$  be a current on  $\tilde{M}$  satisfying Theorem 3.9 (i). Let  $f^*: V^{1,1}(M)_{\mathbb{R}} \rightarrow V^{1,1}(\tilde{M})_{\mathbb{R}}$  be the natural map. Then  $f^*([T]) = [\tilde{T}]$ .*

*Proof.* By condition (B), there exist  $d$ -closed smooth real  $(1, 1)$ -forms  $\varphi$  and  $\tilde{\varphi}$  such that  $\beta([\varphi]) = [T]$  and  $\tilde{\beta}([\tilde{\varphi}]) = \tilde{T}$ ; i.e.,  $T = \varphi + \bar{\partial}S + \partial\bar{S}$  and  $\tilde{T} = \tilde{\varphi} + \bar{\partial}R + \partial\bar{R}$  for suitable currents  $S$  and  $R$ . Therefore

$$f_*\tilde{\varphi} - \varphi = \bar{\partial}(S - f_*R) + \partial(\bar{S} - f_*\bar{R}),$$

but  $\beta$  is injective; hence  $f_*\tilde{\varphi} - \varphi$  is  $d$ -exact. Using the results on the link between the cohomology rings of  $M, \tilde{M}, E$  and  $Y$  (see, e.g., [7, p. 285]) we get

$$\tilde{\varphi} - f^*\varphi = \sum_\alpha c_\alpha[E_\alpha] + dQ$$

for a suitable current  $Q$ .

Recall that there exist open sets  $V$  such that  $\tilde{T}|_{f^{-1}(V)}$  is the limit of components of boundaries; hence the same holds for  $\tilde{\varphi}$ . Moreover

we can choose  $V$  such that  $H^2(V, \mathbf{R}) = 0$ , so that  $\varphi|_V = d\psi$ . Thus  $\sum_{\alpha} c_{\alpha}[E_{\alpha}] = \tilde{\varphi} - f^*\varphi - dQ$  is the limit of components of boundaries in  $f^{-1}(V)$ ; by Proposition 2.7,  $c_{\alpha} = 0 \ \forall \alpha$ , so that

$$f^*([T]) = f^*\beta([\varphi]) = \tilde{\beta}f^*([\varphi]) = \tilde{\beta}f^*([\varphi]) = \tilde{\beta}([\tilde{\varphi}]) = \tilde{T}.$$

#### 4. The main theorem

In this section we use the machinery developed in the previous sections to get some metric results. We start with a lemma concerning condition (B).

**4.1. Lemma.** *Let  $f: \tilde{M} \rightarrow M$  be a proper modification. If  $\tilde{M}$  satisfies condition (B) (that is,  $\beta: H^{1,1}(M, \mathbf{R}) \rightarrow V^{1,1}(M)_{\mathbf{R}}$  is injective and  $\text{Im } \beta$  contains all positive elements of  $V^{1,1}(M)_{\mathbf{R}}$ ), then  $M$  also satisfies (B).*

*Proof.* Let us consider the following commutative diagram (see (1.3)):

$$\begin{array}{ccc} H^{1,1}(M, \mathbf{R}) & \xrightarrow{\beta} & V^{1,1}(M)_{\mathbf{R}} \\ \downarrow f^* & & \downarrow f^* \\ H^{1,1}(\tilde{M}, \mathbf{R}) & \xrightarrow{\tilde{\beta}} & V^{1,1}(\tilde{M})_{\mathbf{R}} \\ \downarrow f_* & & \downarrow f_* \\ H^{1,1}(M, \mathbf{R}) & \xrightarrow{\beta} & V^{1,1}(M)_{\mathbf{R}} \end{array}$$

where  $f_* \circ f^*$  is the identity. Denote by  $[ \ ]$  the classes in all groups that appear in the diagram. By hypothesis,  $\tilde{\beta}$  is injective; hence  $\beta$  is injective too. Let  $T$  be a positive  $\partial\bar{\partial}$ -closed current on  $M$  of degree  $(1, 1)$ , and  $\tilde{T}$  be a positive  $\partial\bar{\partial}$ -closed current on  $\tilde{M}$  of degree  $(1, 1)$  given by Theorem 3.9. We know that there exists a  $d$ -closed form  $\psi \in \mathcal{E}^{1,1}(\tilde{M})_{\mathbf{R}}$  such that  $\tilde{\beta}([\psi]) = [\tilde{T}]$ . Therefore

$$\beta f_*([\psi]) = f_*\tilde{\beta}([\psi]) = [f_*\tilde{T}] = [T]. \quad \text{q.e.d.}$$

Now we can state and prove the Main Theorem. Here the manifolds are supposed to be compact, since Theorem 1.2 is needed.

**4.2. Main Theorem.** *Let  $M, \tilde{M}$  be compact complex manifolds, and  $f: \tilde{M} \rightarrow M$  be a modification. If  $\tilde{M}$  is balanced and satisfies (B) (in particular, if it is Kähler), then  $M$  is balanced and satisfies (B) too.*

*Proof.* (B) holds for  $M$  by Lemma 4.1.

Let  $T = \bar{\partial}S + \partial\bar{S}$  be a positive current of degree  $(1, 1)$  on  $M$ . If we prove that  $T = 0$ , we get the thesis by Theorem 1.2. Let  $\tilde{T}$  be given



by Theorem 3.9; then  $f^*([T]) = [\tilde{T}]$  by Proposition 3.10. Hence  $\tilde{T}$  is a positive component of a boundary on a balanced manifold. This implies  $\tilde{T} = 0$ . Thus  $\text{supp } T \subseteq Y$ , but the codimension of  $Y$  is greater than one; hence by Theorem 2.1,  $T = 0$ .

**Remark.** In the proof of the previous theorem, problems arising from changing charts are avoided. As a matter of fact, one may hope to prove the Main Theorem directly. Starting from a strictly positive  $(1, 1)$ -form  $\tilde{\omega}$  on  $\tilde{M}$  with  $d\tilde{\omega}^{N-1} = 0$ , try to construct an analogous form  $\omega$  on  $M$ . But obviously we cannot hope that  $\omega|_{M-Y} = f_*\tilde{\omega}|_{M-Y}$ , because  $f_*\tilde{\omega}$  “blows up” near  $Y$ . So we should modify  $f_*\tilde{\omega}$  on coordinate open sets which meet  $Y$ , and then glue together these currents. This seems to be much more complicated than our procedure, which consists basically in extending the current  $T$  from  $\tilde{M} - E$  to  $\tilde{M}$ .

Theorem 4.2 has some interesting corollaries; to state them let us recall the definition of the class  $\mathcal{E}$  of Fujiki [5, p. 34–35].

**4.3. Definition.** A reduced (compact) complex analytic space  $X$  belongs to  $\mathcal{E}$  if it is a meromorphic image of a compact Kähler space.

Varouchas proved that  $\mathcal{E}$  is nothing but the class of spaces bimeromorphic to some compact Kähler manifold:

**4.4. Theorem** [22, Theorem 3]. *If  $X \in \mathcal{E}$ , then there exist a compact Kähler manifold  $K$  and a modification  $f: K \rightarrow X$ .*

So we get by Theorem 4.2 a result about “nice” hermitian metrics on manifolds in the class  $\mathcal{E}$ .

**4.5. Corollary.** *Every manifold in the class  $\mathcal{E}$  is balanced.*

And in particular we have

**4.6. Corollary.** *Moishezon manifolds are balanced.*

Notice that there exist compact balanced manifolds not in the class  $\mathcal{E}$ , e.g., the Iwasawa manifold  $I_3$ .

In order to study metrics in connection with modifications of balanced manifolds, let us start from a more general situation and introduce the following definition.

**4.7. Definition.** A complex manifold  $M$  is called *p-Kähler* if it carries a strictly weakly positive smooth closed  $(p, p)$ -form, called a *p-Kähler form*.

For more details about this subject see [2], here we may only point out that 1-Kähler is equivalent to Kähler and  $(N - 1)$ -Kähler is equivalent to balanced.

**4.8. Theorem.** *Let  $M$  and  $\tilde{M}$  be compact p-Kähler manifolds, let  $f: \tilde{M} \rightarrow M$  be a proper modification, and call  $Y$  the degeneracy set, with  $p > \dim Y$ . For every p-Kähler form  $\Omega$  on  $M$ , there exists a*

$p$ -Kähler form  $\tilde{\Omega}$  on  $\tilde{M}$  such that  $[\Omega] = [f_*\tilde{\Omega}]$  in the  $(p, p)$ -Aeppli group  $\Lambda^{p,p}(M)_{\mathbb{R}}$ .

*Proof.* Let  $\Omega$  and  $\Omega'$  be  $p$ -Kähler forms on  $M$  and  $\tilde{M}$  respectively. Since  $p > \dim Y$ , arguing as in [22, pp. 251–252], we get an open neighborhood  $U$  of  $Y$  in  $M$  and a real current  $R$  in  $U$  such that

$$f_*\Omega' = i\partial\bar{\partial}R.$$

Since  $f_*\Omega'$  is smooth and  $\partial\bar{\partial}$ -exact in  $U - Y$ , there exists a smooth real  $(p-1, p-1)$ -form  $\beta$  in  $U - Y$  such that

$$f_*\Omega' |_{U-Y} = i\partial\bar{\partial}\beta.$$

Moreover,  $R - \beta = \gamma + \bar{\partial}C + \partial\bar{C}$  in  $U - Y$ , for a suitable smooth real  $\partial\bar{\partial}$ -closed  $(p-1, p-1)$ -form  $\gamma$  and a current  $C$ .

Now choose an open set  $W$  such that  $Y \subset W \Subset U$  and a real function  $g \in \mathcal{C}_0^\infty(U)$ , with  $g = 1$  in  $W$ ; take

$$D := g(\beta + \gamma) + \bar{\partial}(gC) + \partial(g\bar{C}).$$

Since  $i\partial\bar{\partial}D$  is smooth in  $M - Y$ ,  $\chi := (f|_{\tilde{M}-E})^*i\partial\bar{\partial}D$  is a smooth  $(p, p)$ -form in  $\tilde{M} - E$  which coincides with  $\Omega'$  in  $f^{-1}(W) - E$ ; therefore  $\chi$  can be extended to a real smooth  $(p, p)$ -form on the whole of  $\tilde{M}$ , which is supported in  $f^{-1}(U)$  and strictly weakly positive in  $f^{-1}(W)$ . Choose  $\varepsilon > 0$  such that

$$\tilde{\Omega} := f^*\Omega + \varepsilon\chi$$

is strictly weakly positive on  $\tilde{M}$ , so we get  $\Omega - f_*\tilde{\Omega} = i\partial\bar{\partial}\varepsilon D$ . q.e.d.

In [2] we studied a kind of modification for which the hypotheses of Theorem 4.8 hold, so that we have now some new information about the link between  $p$ -Kähler forms on  $M$  and  $\tilde{M}$ . Moreover for  $p = N - 1$  we can give a metric interpretation:

**4.9. Corollary.** *Let  $M$  and  $\tilde{M}$  be compact balanced manifolds and  $f: \tilde{M} \rightarrow M$  a modification. For every balanced metric  $h$  on  $M$  with Kähler form  $\omega$  there exists a balanced metric  $\tilde{h}$  on  $\tilde{M}$  with Kähler form  $\tilde{\omega}$  such that  $\omega^{N-1} - f_*\tilde{\omega}^{N-1}$  is a  $\partial\bar{\partial}$ -exact current.*

## References

- [1] L. Alessandrini & G. Bassanelli, *A balanced proper modification of  $P_3$* , Comment. Math. Helv. **66** (1991) 505–511.
- [2] —, *Positive  $\partial\bar{\partial}$ -closed currents and non-Kähler geometry*, J. Geometric Analysis **3** (1992).
- [3] —, *Smooth proper modifications of compact Kähler manifolds*, Proc. Internat. Workshop on Complex Analysis (Wuppertal 1990), Complex Analysis, Aspects of Mathematics E, Vol. 17, Vieweg, Germany, 1991.

- [4] A. Blanchard, *Les variétés analytiques complexes*, Ann. Sci. Ecole Norm. Sup. **73** (1958) 157–202.
- [5] A. Fujiki, *Closedness of the Douady spaces of compact Kähler spaces*, Publ. Res. Inst. Math. Sci. **14** (1978) 1–52.
- [6] P. Gauduchon, *Fibrés hermitiens a endomorphisme de Ricci non négatif*, Bull. Soc. Math. France **105** (1977) 113–140.
- [7] H. Grauert & O. Riemenschneider, *Verschwindungssätze für analytische Kohomologiegruppen auf komplexen Räumen*, Invent. Math. **11** (1970) 263–292.
- [8] P. Griffiths & J. Harris, *Principles of algebraic geometry*, Wiley Interscience, New York, 1978.
- [9] R. Hartshorne, *Algebraic geometry*, Springer, New York, 1977.
- [10] R. Harvey, *Holomorphic chains and their boundaries*, Proc. Sympos. Pure Math., Vol. 30, Part 1, Amer. Math. Soc., Providence, RI, 1977, 309–382.
- [11] R. Harvey & J. R. Lawson, *An intrinsic characterization of Kähler manifolds*, Invent. Math. **74** (1983) 169–198.
- [12] H. Hironaka, *Flattening theorems in complex analytic geometry*, Amer. J. Math. **97** (1975) 503–547.
- [13] H. Hironaka & H. Rossi, *On the equivalence of embeddings of exceptional complex spaces*, Math. Ann. **156** (1964) 313–333.
- [14] L. Hörmander, *The analysis of linear partial differential operators*, I, Grundlehren Math. Wiss., no. 256, Springer, Berlin, 1983.
- [15] P. Lelong, *Plurisubharmonic functions and positive differential forms*, Gordon and Breach, New York, 1969.
- [16] M. L. Michelson, *On the existence of special metrics in complex geometry*, Acta Math. **143** (1983) 261–295.
- [17] I. R. Shafarevich, *Basic algebraic geometry*, Springer, Berlin, 1977.
- [18] B. Shiffman, *Extension of positive line bundles and meromorphic maps*, Invent. Math. **15** (1972) 332–347.
- [19] Y. T. Siu, *Analyticity of sets associated to Lelong numbers and the extension of closed positive currents*, Invent. Math. **27** (1974) 53–156.
- [20] H. Skoda, *Prolongement des courants, positif, fermés de masse finie*, Invent. Math. **66** (1982) 361–376.
- [21] J. Varouchas, *Propriétés cohomologiques d'une classe de variétés analytiques complexes compactes*, Sem. d'Analyse Lelong-Dolbeault-Skoda 1983-84, Lecture Notes in Math., Vol. 1198, Springer, Berlin, 1985, 233–243.
- [22] —, *Sur l'image d'une variété kählérienne compacte*, Seminaire Norguet 1983-84, Lecture Notes in Math., Vol. 1188, Springer, Berlin, 1985, 245–259.
- [23] R. O. Wells, *Moishezon spaces and Kodaira embedding theorem*, Proc. Tulane Univ. Program on Value Distribution Theory in Complex Analysis and Related Topics in Differential Geometry, 1972-73, Dekker, New York, 1974, 29–42.

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